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Two-boson realizations of the polynomial angular momentum algebra and some applications

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In this paper two kinds of two-boson realizations of the polynomial angular momentum algebra are obtained by generalizing the well known Jordan–Schwinger realizations of the SU(2) and SU(1,1) algebras. Especially, for the Higgs algebra, an unitary realization and two nonunitary realizations, together with the properties of their respective acting spaces are discussed in detail. Furthermore, similarity transformations, which connect the nonunitary realizations with the unitary ones, are gained by solving the corresponding unitarization equations. As applications, the dynamical symmetry of the Kepler system in a two-dimensional curved space is studied and phase operators of the Higgs algebra are constructed.

KEY WORDS: polynomial angular momentum algebra, Higgs algebra, boson realization, Kepler system, phase operator

AMS subject classification: 81R

1. Introduction

The boson realization (or boson expansion) of Lie algebra has played a central role in the study of algebraic models for atomic [1], nuclear [2], and molecular structures [3,4]. One of the most famous cases is the Jordan–Schwinger realization of angular momentum in quantum mechanics, which, corresponding to the Lie algebra SU(2) or SO(3) [5], may be described by means of the occupation number representation of the two-dimensional isotropic harmonic oscillator [6].

In recent years, the polynomial angular momentum algebra (PAMA) and its increasing applications in quantum problems have been the focus of very active research. This kind of PAMA, spanned by three elements $\mathcal{J}_\mu$ ($\mu = +, -, 3$), has a coset structure $h + v$ [7], where h is the Lie algebra U(1) generated by $\mathcal{J}_3$; the remaining two elements $\mathcal{J}_+, \mathcal{J}_- \in v$ transform according to a representation of U(1), and their commutator yields a polynomial function of order n in the operator $\mathcal{J}_3 \in$ U(1), i.e.,

$$[\mathcal{J}_3, \mathcal{J}_\pm] = \pm\mathcal{J}_\pm, \quad [\mathcal{J}_+, \mathcal{J}_-] = \sum_{i=0}^{n} C_i(\mathcal{J}_3)^i, \tag{1}$$

where the coefficients C_i ($i = 0, 1, \ldots, n$) are real constants. When $C_1 = 2$ (or -2) and $C_0 = C_j = 0$ ($j \geqslant 2$), equation (1) goes back to the commutation relations satisfied by the angular momentum algebra SU(2) (or its noncompact type SU(1,1)). Hence, the PAMA can be viewed as a type of polynomial deformation of SU(2) (or SU(1,1)), or a type of nonlinear extension of U(1).

The first special case of the PAMA is the so-called Higgs algebra, which, here denoted by $\mathcal{H}$, was used by Higgs [8] to establish the existence of additional symmetries for the isotropic oscillator and Kepler potentials in a two-dimensional curved space. Later, Zhedanov [9] presented a connection between the Higgs algebra $\mathcal{H}$ and the quantum group $SU_q(2)$. [10] Daskaloyannis [11] and Bonatsos *et al.* [12,13] discussed the PAMA by means of the generalized deformed oscillator, respectively, and Quesne [14] related it to the generalized deformed parafermion. Junker and Roy [15] constructed the (nonlinear) coherent states of $\mathcal{H}$ for the conditionally exactly solvable model with the radial potential of harmonic oscillator, and Sunilkumar *et al.* [16] did for the quadrilinear boson Hamiltonian describing four-photon process and showed [17] that the PAMA of order (n_1+n_2+1) may be constructed by combining two given mutually commuting PAMAs with their respective orders being n_1 and n_2. Recently, Beckers *et al.* [18] and Debergh [19] realized $\mathcal{H}$, which is seen as a spectrum generating algebra in their method, by single-variable differential operators in the study of (quasi-) exactly solvable problems, and also construct a special unitary two-boson realization to study the Karassiov–Klimov Hamiltonian in the quantum optics. Ruan *et al.* [20] studied indecomposable representations of the PAMA of quadratic type, and then from these representations obtained its inhomogeneous boson realizations. In the present work we will study in detail for PAMA two-boson realizations, which are analogous to the well known Jordan–Schwinger realizations of the SU(2) and SU(1,1) algebras [5], and some applications. In order to obtain the explicit results, we shall restrict ourself to the Higgs algebra $\mathcal{H}$.

This paper is arranged as follows. In section 2, some elementary results of the Jordan–Schwinger realizations of SU(2) and SU(1,1) and of the irreducible unitary representations of $\mathcal{H}$ are briefly reviewed, respectively. In section 3, two kinds of two-boson realizations of $\mathcal{H}$ are studied in detail, such as the unitary realizations, the nonunitary realizations, and their respective acting spaces. In section 4, we first discuss generally the unitarization equations satisfied by the nonunitary realizations, then calculate the explicit expressions for the corresponding similarity transformations, which may relate the nonunitary realizations to the unitary ones. In section 5, as applications, by making use of the results obtained in section 3, the dynamical symmetry of the Kepler system in the two-dimensional curved space is studied and the phase operators of $\mathcal{H}$ are constructed. A simple discussion is given in the final section.

2. Notations and some elementary results

In this section, some elementary results, along with notations, to be used later are briefly reviewed, such as the standard Jordan–Schwinger realizations of the SU(2) and SU(1,1) algebras, the irreducible unitary representations of the Higgs algebra $\mathcal{H}$, and so on.

2.1. The Jordan–Schwinger realizations of SU(2) and SU(1,1)

Denote three generators of SU(2) and its noncompact type SU(1,1) by $\{J_+, J_-, J_3\}$, then their commutation relations may be written in a compact form

$$[J_+, J_-] = 2\lambda J_3, \quad [J_3, J_\pm] = \pm J_\pm, \tag{2}$$

where $\lambda = 1$ for SU(2) and $\lambda = -1$ for SU(1,1).

In terms of the Jordan–Schwinger mapping [6], the generators of SU(2) and SU(1,1) may be respectively realized by two pairs of mutually commuting boson operators $\{a_i, a_i^+ | \, i = 1, 2\}$ (the annihilation operators a_i are adjoint to the creation operators a_i^+, i.e., $a_i = (a_i^+)^\dagger$, $a_i^+ = (a_i)^\dagger$) as

$$\begin{aligned} J_+ &= a_1^+ a_2, \\ J_- &= a_1 a_2^+, \\ J_3 &= \tfrac{1}{2}(\hat{n}_1 - \hat{n}_2) \end{aligned} \tag{3}$$

for SU(2), and

$$\begin{aligned} J_+ &= a_1^+ a_2^+, \\ J_- &= a_1 a_2, \\ J_3 &= \tfrac{1}{2}(\hat{n}_1 + \hat{n}_2 + 1) \end{aligned} \tag{4}$$

for SU(1,1), where $\hat{n}_i \equiv a_i^+ a_i$ $(i = 1, 2)$ are the corresponding particle number operators, which, together with the boson operators $\{a_i, a_i^+\}$, satisfy the commutation relations

$$\begin{aligned} [a_i, a_j^+] &= \delta_{ij}, \\ [\hat{n}_i, a_j^+] &= \delta_{ij} a_j^+, \\ [\hat{n}_i, a_j] &= -\delta_{ij} a_j. \end{aligned} \tag{5}$$

Furthermore, the complete set of basis vectors of Fock space, $\mathcal{F} \equiv \{|n_1 n_2\rangle|\ n_1, n_2 = 0, 1, 2, \dots\}$, may be constructed from the vacuum state $|00\rangle$ of the two-dimensional harmonic oscillator by using the definition

$$|n_1 n_2\rangle = \frac{(a_1^+)^{n_1}(a_2^+)^{n_2}}{\sqrt{n_1! n_2!}}|00\rangle. \tag{6}$$

In fact, these vectors are the common normalized eigenvectors of $\hat{n}_1$ and $\hat{n}_2$ belonging to eigenvalues n_1 and n_2 respectively, i.e.,

$$\hat{n}_i|\dots n_i \dots\rangle = n_i|\dots n_i \dots\rangle, \quad i = 1,\ 2, \tag{7}$$

and satisfy

$$\begin{aligned} a_i|\dots n_i \dots\rangle &= \sqrt{n_i}|\dots n_i - 1 \dots\rangle, \\ a_i^+|\dots n_i \dots\rangle &= \sqrt{n_i + 1}|\dots n_i + 1 \dots\rangle. \end{aligned} \tag{8}$$

Correspondingly, the common eigenvectors $|jm\rangle$ of the angular momentum operators $\mathbf{J}^2$ and J_3 may also be expressed in the Jordan–Schwinger representation as

$$|jm\rangle = \frac{(a_1^+)^{j+m}(a_2^+)^{j-m}}{\sqrt{(j+m)!(j-m)!}}|00\rangle. \tag{9}$$

Comparison between equations (9) and (6) leads immediately to

$$\hat{n}_i|jm\rangle = [j - (-1)^i m]|jm\rangle, \quad i = 1,\ 2, \tag{10}$$

that is, the quantum numbers n_1 and n_2 are related to j and m by the equations $n_1 = j + m$ and $n_2 = j - m$.

2.2. The Higgs algebra $\mathcal{H}$ and its irreducible unitary representation

Taking $C_2 = C_j = 0$ ($j > 3$) in equation (1), it follows that the three generators $\{\mathcal{J}_\pm, \mathcal{J}_3\}$ of the Higgs algebra $\mathcal{H}$ satisfy the following commutation relations

$$[\mathcal{J}_3, \mathcal{J}_\pm] = \pm\mathcal{J}_\pm, \quad [\mathcal{J}_+, \mathcal{J}_-] = C_1\mathcal{J}_3 + C_3\mathcal{J}_3^3. \tag{11}$$

In analogy with SU(2) [5], the Casimir invariant of $\mathcal{H}$ reads

$$\mathcal{C} = \tfrac{1}{2}(\mathcal{J}_+\mathcal{J}_- + \mathcal{J}_-\mathcal{J}_+) + \left(\tfrac{1}{2}C_1 + \tfrac{1}{4}C_3\right)\mathcal{J}_3^2 + \tfrac{1}{4}C_3\mathcal{J}_3^4, \tag{12}$$

which commutes with the three generators of $\mathcal{H}$, i.e.,

$$[\mathcal{C}, \mathcal{J}_\pm] = [\mathcal{C}, \mathcal{J}_3] = 0. \tag{13}$$

It is worthy of reminding the readers that the constant C_1 in equation (11) is remained for convenience though it may become some fixed real number, say q, by rescaling the generators, $\mathcal{J}_\pm \to \sqrt{q/C_1}\mathcal{J}_\pm$.

Making use of the parallel treatment of angular momentum in quantum mechanics [5], it is not difficult to obtain the following unitary representation of $\mathcal{H}$ in the common eigenvectors $|\tilde{j}\tilde{m}\rangle$ of the elements $\{\mathcal{C}, \mathcal{J}_3\}$, with $\tilde{j}$ and $\tilde{m}$ labelling the eigenvalues of $\mathcal{C}$ and $\mathcal{J}_3$, respectively, [18,21]

$$\begin{aligned}
\langle \tilde{j}\tilde{m}+1|\mathcal{J}_+|\tilde{j}\tilde{m}\rangle &= \sqrt{\frac{1}{2}C_1[\tilde{j}(\tilde{j}+1)-\tilde{m}(\tilde{m}+1)]+\frac{1}{4}C_3[\tilde{j}^2(\tilde{j}+1)^2-\tilde{m}^2(\tilde{m}+1)^2]},\\
\langle \tilde{j}\tilde{m}-1|\mathcal{J}_-|\tilde{j}\tilde{m}\rangle &= \sqrt{\frac{1}{2}C_1[\tilde{j}(\tilde{j}+1)-\tilde{m}(\tilde{m}-1)]+\frac{1}{4}C_3[\tilde{j}^2(\tilde{j}+1)^2-\tilde{m}^2(\tilde{m}-1)^2]},\\
\langle \tilde{j}\tilde{m}|\mathcal{J}_3|\tilde{j}\tilde{m}\rangle &= \tilde{m},\\
\langle \tilde{j}\tilde{m}|\mathcal{C}|\tilde{j}\tilde{m}\rangle &= \frac{1}{2}C_1\tilde{j}(\tilde{j}+1)+\frac{1}{4}C_3\tilde{j}^2(\tilde{j}+1)^2.
\end{aligned} \tag{14}$$

Here we have adopted the same phase factor as the Condon–Shortley convention of SU(2) so that the matrix elements of $\mathcal{J}_\pm$ are real. In equation (14), $\tilde{j}$ may take half-integers, i.e., 0, 1/2, 1, 3/2, ... , and for the finite dimensional representation with a fixed $\tilde{j}$, the values that m may take, being a part of $\{-\tilde{j}, -\tilde{j}+1, \ldots, \tilde{j}\}$, are different for different C_1's and C_3's [21].

3. Two kinds of two-boson realizations of $\mathcal{H}$

In this section, we will study two kinds of two-boson realizations of $\mathcal{H}$, which are analogous to the Jordan–Schwinger realizations of SU(2) and SU(1,1), respectively.

3.1. The first kind of realizations

The Jordan–Schwinger realization (3) of SU(2) reminds us that the first kind of two-boson realizations of $\mathcal{H}$ may be chosen in the following form

$$\begin{aligned}
\dot{B}^{(k,l)}(\mathcal{J}_+) &= \dot{f}(\hat{n}_1,\hat{n}_2)(a_1^+)^k a_2^l,\\
\dot{B}^{(k,l)}(\mathcal{J}_-) &= a_1^k(a_2^+)^l\dot{g}(\hat{n}_1,\hat{n}_2),\\
\dot{B}^{(k,l)}(\mathcal{J}_3) &= \dot{h}(\hat{n}_1,\hat{n}_2),
\end{aligned} \tag{15}$$

where k and l are positive integers, $\dot{f}(\hat{n}_1,\hat{n}_2)$, $\dot{g}(\hat{n}_1,\hat{n}_2)$ and $\dot{h}(\hat{n}_1,\hat{n}_2)$, being the operator functions of $\hat{n}_1$ and $\hat{n}_2$, have to be determined by the commutation relations (11) of $\mathcal{H}$. For a fixed (k,l), the action of $\dot{B}^{(k,l)}(\mathcal{J}_\pm)$ on some basis vector $|n_1n_2\rangle$ of the boson Fock space $\mathcal{F}$ gives another basis vector $|n_1 \pm k, n_2 \mp l\rangle$.

The first equation of equation (11) requires that $\dot{h}(\hat{n}_1,\hat{n}_2)$ satisfies the simple two-variable difference equation

$$\dot{h}(\hat{n}_1, \hat{n}_2) - \dot{h}(\hat{n}_1 - k, \hat{n}_2 + l) = 1. \tag{16}$$

Its solution reads

$$\dot{h}(\hat{n}_1, \hat{n}_2) = \frac{\hat{n}_1}{2k} - \frac{\hat{n}_2}{2l} + \alpha, \tag{17}$$

here α, being a real constant, needs further determining by considering the irreducible representation of $\mathcal{H}$ given in section 2.2. Equation (17) clearly shows that the two-boson realization (15) can not be reduced to the single-boson case by setting $k = 0$ or $l = 0$ because of singularity.

Substituting equation (17) into equation (15), thus, satisfaction of the second equation of equation (11) requires that $\dot{f}(\hat{n}_1, \hat{n}_2)\dot{g}(\hat{n}_1, \hat{n}_2)$ satisfies the following two-variable difference equation

$$\begin{aligned}
&\left[\prod_{i=1}^{k}(\hat{n}_1 - i + 1)\right]\left[\prod_{i=1}^{l}(\hat{n}_2 + i)\right] \dot{f}^{(k,l)}(\hat{n}_1, \hat{n}_2)\dot{g}^{(k,l)}(\hat{n}_1, \hat{n}_2) \\
&\quad - \left[\prod_{i=1}^{k}(\hat{n}_1 + i)\right]\left[\prod_{i=1}^{l}(\hat{n}_2 - i + 1)\right] \dot{f}^{(k,l)}(\hat{n}_1 + k, \hat{n}_2 - l)\dot{g}^{(k,l)}(\hat{n}_1 + k, \hat{n}_2 - l) \\
&= C_1\left(\frac{\hat{n}_1}{2k} - \frac{\hat{n}_2}{2l} + \alpha\right) + C_3\left(\frac{\hat{n}_1}{2k} - \frac{\hat{n}_2}{2l} + \alpha\right)^3.
\end{aligned} \tag{18}$$

In the process of obtaining the above equation, we have used the fundamental relations

$$\begin{aligned}
a_i^k f(\ldots \hat{n}_i \ldots) &= f(\ldots, \hat{n}_i + k, \ldots) a_i^k, \quad i = 1,\ 2, \\
(a_i^+)^k f(\ldots \hat{n}_i \ldots) &= f(\ldots, \hat{n}_i - k, \ldots)(a_i^+)^k,
\end{aligned} \tag{19}$$

which follow from equation (5) for any function $f(\ldots \hat{n}_i \ldots)$.

Note that equation (18) only fixes the product $\dot{f}(\hat{n}_1, \hat{n}_2)\dot{g}(\hat{n}_1, \hat{n}_2)$. Different choices of the two functions, as well as the constant α, may produce a variety of realizations for $\mathcal{H}$. However, it is very difficult to obtain the general solutions of equation (18) for arbitrary k and l. Below will study in more detail the special case of $(k, l) = (1, 1)$, and give directly the results of the case of $(k, l) = (2, 2)$.

1. The (1,1) case.

Inserting $k = l = 1$ into equation (18) and solving it, we may obtain the following two solutions

$$\begin{aligned}
\dot{f}_1^{(1,1)}(\hat{n}_1, \hat{n}_2)\dot{g}_1^{(1,1)}(\hat{n}_1, \hat{n}_2) = \frac{1}{8\hat{n}_1}(\hat{n}_1 + 2\alpha)\{4C_1 + C_3[\hat{n}_1(\hat{n}_1 + 4\alpha) \\
+(\hat{n}_2 + 1)^2 + (2\alpha + 1)(2\alpha - 1)]\},
\end{aligned} \tag{20}$$

and

$$\dot{f}_2^{(1,1)}(\hat{n}_1,\hat{n}_2)\dot{g}_2^{(1,1)}(\hat{n}_1,\hat{n}_2) = \frac{1}{8(\hat{n}_2+1)}(\hat{n}_2-2\alpha+1)\{4C_1+C_3[\hat{n}_1^2 + \hat{n}_2(\hat{n}_2-4\alpha+2)+4\alpha(\alpha-1)]\}. \tag{21}$$

From them we have some freedom in the choice of the functions $\dot{f}_i^{(1,1)}(\hat{n}_1,\hat{n}_2)$ ($i = 1$, 2) and $\dot{g}_i^{(1,1)}(\hat{n}_1,\hat{n}_2)$. However here we need only consider the first solution (20) because of the symmetry between the solutions (20) and (21)

$$\hat{n}_1 \leftrightarrow \hat{n}_2+1 \quad \text{and} \quad \alpha \leftrightarrow -\alpha.$$

(1) If the unitary relations need satisfying, i.e.,

$$\dot{B}^{(1,1)}(\mathcal{J}_\pm) = \left(\dot{B}^{(1,1)}(\mathcal{J}_\mp)\right)^\dagger, \tag{22}$$

($\dot{B}^{(1,1)}(\mathcal{J}_3)$ is already hermitian), which lead to $\dot{f}_1^{(1,1)}(\hat{n}_1,\hat{n}_2) = \dot{g}_1^{(1,1)}(\hat{n}_1,\hat{n}_2)$, then solving equation (20) and substituting the expression of $\dot{f}_1^{(1,1)}(\hat{n}_1,\hat{n}_2)$ into equation (15), we may obtain

$$\begin{aligned}
\dot{B}_1^{(1,1)}(\mathcal{J}_+) &= \left\{\tfrac{1}{8\hat{n}_1}(\hat{n}_1+2\alpha)\{4C_1+C_3[\hat{n}_1(\hat{n}_1+4\alpha) + (\hat{n}_2+1)^2+(2\alpha+1)(2\alpha-1)]\}\right\}^{1/2} a_1^+ a_2,\\
\dot{B}_1^{(1,1)}(\mathcal{J}_-) &= a_1 a_2^+ \left\{\tfrac{1}{8\hat{n}_1}(\hat{n}_1+2\alpha)\{4C_1+C_3[\hat{n}_1(\hat{n}_1+4\alpha) + (\hat{n}_2+1)^2+(2\alpha+1)(2\alpha-1)]\}\right\}^{1/2},\\
\dot{B}_1^{(1,1)}(\mathcal{J}_3) &= \tfrac{1}{2}(\hat{n}_1-\hat{n}_2)+\alpha.
\end{aligned} \tag{23}$$

It can be easily checked that the realization (23) satisfies equation (11) for arbitrary α.

Inserting equation (23) into equation (12), the Casimir invariant $\mathcal{C}$ of $\mathcal{H}$ may be expressed in terms of the boson number operators $\hat{n}_1$ and $\hat{n}_2$ as

$$\mathcal{C} = \tfrac{1}{64}(\hat{N}+2\alpha)(\hat{N}+2\alpha+2)[8C_1+C_3(\hat{N}+2\alpha)(\hat{N}+2\alpha+2)], \tag{24}$$

where $\hat{N} = \hat{n}_1+\hat{n}_2$ is the total boson number operator. The equation (24) shows clearly that $\mathcal{C}$ depends only on $\hat{N}$.

Calculating the expectation value $\langle n_1 n_2|\mathcal{C}|n_1 n_2\rangle$ and comparing it with the fourth equation of equation (14), we have

$$\tilde{j} = \tfrac{1}{2}(N+2\alpha). \tag{25}$$

The fact that the values of $\tilde{j}$ are half integers ($\tilde{j} = 0,\ 1/2,\ 1,\dots$) requires $\alpha = 0$, thus, the irreducible representation $\tilde{j}$ of $\mathcal{H}$ is characterized by the total boson number N, namely, $\tilde{j} = N/2$. The similar conclusion exists for SU(2) [5]. Correspondingly, equation (23) leads to the simplest form

$$\begin{aligned}
&\dot{B}_2^{(1,1)}(\mathcal{J}_+) = \sqrt{\tfrac{1}{2}C_1 + \tfrac{1}{8}C_3[\hat{n}_1^2 + \hat{n}_2(\hat{n}_2+2)]}\,a_1^+ a_2,\\
&\dot{B}_2^{(1,1)}(\mathcal{J}_-) = a_1 a_2^+ \sqrt{\tfrac{1}{2}C_1 + \tfrac{1}{8}C_3[\hat{n}_1^2 + \hat{n}_2(\hat{n}_2+2)]},\\
&\dot{B}_2^{(1,1)}(\mathcal{J}_3) = \tfrac{1}{2}(\hat{n}_1 - \hat{n}_2),
\end{aligned} \tag{26}$$

which may also be obtained by considering the second solution (21) with setting $\alpha = 0$. When $C_1 = 2$ and $C_3 = 0$, equation (26) becomes the standard Jordan–Schwinger realization (3) of SU(2).

Now discuss the properties of the spaces that $\dot{B}_2^{(1,1)}(\mathcal{J}_\mu)$ ($\mu = \pm$, 3) act on. We observe that for $C_3 \neq 0$ the square-root symbols appear in the two-boson realization (26), which is analogous to the Holstein–Primakoff single-boson realization of SU(2) [22]. The acting spaces of $\dot{B}_2^{(1,1)}(\mathcal{J}_\mu)$ may be certain subspaces of the Fock space $\mathcal{F} = \{|n_1 n_2\rangle \mid n_1, n_2 = 0, 1, 2, \ldots\}$, in which n_1 and n_2 need limiting in order that the values of the square roots appeared in the matrix elements $\langle n_1 \pm 1 n_2 \mp 1|\dot{B}_2^{(1,1)}(\mathcal{J}_\pm)|n_1 n_2\rangle$ must be greater than or equal to zero. For the realization (26), n_1 and n_2 have to satisfy the constraint conditions

$$\begin{cases} (n_1+1)^2 + n_2^2 \geqslant 1 - 4\frac{C_1}{C_3},\\ n_1^2 + (n_2+1)^2 \geqslant 1 - 4\frac{C_1}{C_3}. \end{cases} \tag{27}$$

The results of equation (27), which are pertinent to the relative signs of C_1 and C_3, may be put into the following two categories.

(A) If C_1 has the same sign as C_3, then equation (27) always holds so that the acting space of $\dot{B}_2^{(1,1)}(\mathcal{J}_\mu)$ is the whole Fock space $\mathcal{F}$. In $\mathcal{F}$, the infinite-dimensional nullspaces of $\dot{B}_2^{(1,1)}(\mathcal{J}_+)$ and $\dot{B}_2^{(1,1)}(\mathcal{J}_-)$ are

$$\{|n_1\,0\rangle \mid n_1 = 0,\ 1, \ldots\} \quad \text{and} \quad \{|0\,n_2\rangle \mid n_2 = 0,\ 1, \ldots\},$$

respectively, since they satisfy

$$\dot{B}_2^{(1,1)}(\mathcal{J}_+)\,|n_1\,0\rangle = \dot{B}_2^{(1,1)}(\mathcal{J}_-)\,|0\,n_2\rangle = 0.$$

Obviously, $|00\rangle$ is the common nullspace state of $\dot{B}_2^{(1,1)}(\mathcal{J}_+)$ and $\dot{B}_2^{(1,1)}(\mathcal{J}_-)$

(B) If the sign of C_1 is opposite to that of C_3, then the values of n_1 and n_2 are limited by equation (27). Consider first that n_1 takes independently values, then the smallest value that n_2 may take, which depends on n_1, should be $\zeta_1(n_1) \equiv \left[\sqrt{1 - 4C_1/C_3 - (n_1+1)^2}\right]$, where the symbol $[x]$ for a real number x means taking an integer greater than x, so that the acting space of $\dot{B}_2^{(1,1)}(\mathcal{J}_\mu)$ is

$$\dot{V}_1 = \bigcup_{n_1=0}^{\eta} \dot{V}_1(n_1) \subset \mathcal{F},$$

where

$$\dot{V}_1(n_1) \equiv \{|n_1, \zeta_1(n_1)+i\rangle \mid i=0,\ 1, \ldots\}, \qquad \eta \equiv \left[\sqrt{1-4C_1/C_3}\right]-1.$$

In $\dot{V}_1$, $\dot{V}_1(0)$ is the infinite-dimensional nullspace of $\dot{B}_2^{(1,1)}(\mathcal{J}_-)$ since all the states in $\dot{V}_1(0)$ satisfy $\dot{B}_2^{(1,1)}(\mathcal{J}_-)|0, \zeta_1(0)+i\rangle = 0$ ($i=0,\ 1,\ \ldots$). The subspace $\{|n_1, \zeta_1(n_1)\rangle \mid n_1 = 0, 1, \ldots, \eta\}$ in $\dot{V}_1$ is the $(\eta+1)$-dimensional nullspace of $\dot{B}_2^{(1,1)}(\mathcal{J}_+)$, which satisfies $\dot{B}_2^{(1,1)}(\mathcal{J}_+)|n_1, \zeta_1(n_1)\rangle = 0$. Moreover, $|0, \zeta_1(0)\rangle$ is the common nullspace state of $\dot{B}_2^{(1,1)}(\mathcal{J}_+)$ and $\dot{B}_2^{(1,1)}(\mathcal{J}_-)$.

In view of the simple symmetry $n_1 \leftrightarrow n_2$ between the two equations of equation (27), if n_2 takes independently values, then the smallest value of n_1 should be $\zeta_2(n_2) \equiv \left[\sqrt{1-4C_1/C_3-(n_2+1)^2}\right]$, hence the acting space of $\dot{B}_2^{(1,1)}(\mathcal{J}_\mu)$ is

$$\dot{V}_2 = \bigcup_{n_2=0}^{\eta} \dot{V}_2(n_2) \equiv \bigcup_{n_2=0}^{\eta}\{|\zeta_2(n_2)+i, n_2\rangle \mid i=0,\ 1, \ldots\} \subset \mathcal{F}.$$

In $\dot{V}_2$, $\dot{V}_2(0)$ is the infinite-dimensional nullspace of $\dot{B}_2^{(1,1)}(\mathcal{J}_+)$, $\{|\zeta_2(n_2), n_2\rangle \mid n_2 = 0, 1, \ldots, \eta\}$ is the $(\eta+1)$-dimensional nullspace of $\dot{B}_2^{(1,1)}(\mathcal{J}_-)$, and $|\zeta_2(0), 0\rangle$ is the common nullspace state of $\dot{B}_2^{(1,1)}(\mathcal{J}_+)$ and $\dot{B}_2^{(1,1)}(\mathcal{J}_-)$.

(2) If the unitary relations need not satisfying, it follows from equation (20) that the conventional choice $\dot{g}_1^{(1,1)}(\hat{n}_1, \hat{n}_2) = 1$ (or $\dot{f}_1^{(1,1)}(\hat{n}_1, \hat{n}_2) = 1$) may immediately give rise to a nonunitary two-boson realization

$$\begin{aligned}\dot{B}_3^{(1,1)}(\mathcal{J}_+) &= \tfrac{1}{8\hat{n}_1}(\hat{n}_1+2\alpha)\{4C_1 + C_3[\hat{n}_1(\hat{n}_1+4\alpha)\\ &\quad +(\hat{n}_2+1)^2+(2\alpha+1)(2\alpha-1)]\}a_1^+a_2,\\ \dot{B}_3^{(1,1)}(\mathcal{J}_-) &= a_1a_2^+,\\ \dot{B}_3^{(1,1)}(\mathcal{J}_3) &= \tfrac{1}{2}(\hat{n}_1-\hat{n}_2)+\alpha.\end{aligned} \tag{28}$$

In terms of equation (28), the Casimir invariant $\mathcal{C}$, equation (12), of $\mathcal{H}$ has the same expression as equation (24). So taking $\alpha = 0$ in equation (28) leads to

$$\begin{aligned}\dot{B}_4^{(1,1)}(\mathcal{J}_+) &= \left\{\tfrac{1}{2}C_1 + \tfrac{1}{8}C_3[\hat{n}_1^2 + \hat{n}_2(\hat{n}_2+2)]\right\} a_1^+a_2,\\ \dot{B}_4^{(1,1)}(\mathcal{J}_-) &= a_1a_2^+,\\ \dot{B}_4^{(1,1)}(\mathcal{J}_3) &= \tfrac{1}{2}(\hat{n}_1-\hat{n}_2).\end{aligned} \tag{29}$$

Different from the unitary realization (26), no square-root symbols appear in the above nonunitary realization (29), hence, it may not only avoid the convergence questions associated with the expansion of square-root operator but also make the values of n_1 and n_2 in $\{|n_1n_2\rangle\}$ unlimited, i.e., the acting space of $\dot{B}_4^{(1,1)}(\mathcal{J}_\mu)$ is the whole Fock space. Taking especially $C_1 = 2$ and $C_3 = 0$, equation (29) gives an unitary realization of SU(2), i.e., the Jordan–Schwinger realization (3), while taking $C_1 = -2$ and $C_3 = 0$, equation (29) does a nonunitary realization of SU(1,1). We notice that for $C_3 \neq 0$ the two-boson realization (29) is in fact analogous to the Dyson single-boson realization of SU(2) [23].

(3) Another nonunitary realization may be obtained by choosing $\dot{g}(\hat{n}_1, \hat{n}_2) = \dot{f}(\hat{n}_1 - 1, \hat{n}_2 + 1)$ and $\alpha = 0$ in equation (20) as

$$\begin{aligned} \dot{B}_5^{(1,1)}(\mathcal{J}_+) &= \dot{f}(\hat{n}_1, \hat{n}_2) a_1^+ a_2, \\ \dot{B}_5^{(1,1)}(\mathcal{J}_-) &= \dot{f}(\hat{n}_1, \hat{n}_2) a_1 a_2^+, \\ \dot{B}_5^{(1,1)}(\mathcal{J}_3) &= \tfrac{1}{2}(\hat{n}_1 - \hat{n}_2), \end{aligned} \tag{30}$$

where $\dot{f}(\hat{n}_1, \hat{n}_2)$ satisfies

$$8\dot{f}(\hat{n}_1, \hat{n}_2) = \left\{4C_1 + C_3[\hat{n}_1^2 + \hat{n}_2(\hat{n}_2 + 2)]\right\} \dot{f}^{-1}(\hat{n}_1 - 1, \hat{n}_2 + 1). \tag{31}$$

Note that here $\dot{B}_5^{(1,1)}(\mathcal{J}_\pm) \neq (\dot{B}_5^{(1,1)}(\mathcal{J}_\mp))^\dagger$ for the real function $\dot{f}(\hat{n}_1, \hat{n}_2)$. We call equation (30) a constrained nonunitary realization since $\dot{B}_5^{(1,1)}(\mathcal{J}_+)$ and $\dot{B}_5^{(1,1)}(\mathcal{J}_-)$ utilize the same function $\dot{f}(\hat{n}_1, \hat{n}_2)$. With the help of equation (7), solving equation (31) gives rise to

$$\begin{aligned} \dot{f}(\hat{n}_1, \hat{n}_2) = \exp\Bigg\{(-1)^{\hat{n}_1 - 1}\Bigg[&-\dot{\Omega}_1^{--}(\hat{N}) + \dot{\Omega}_3^{--}(\hat{N}) - \dot{\Omega}_1^{-+}(\hat{N}) + \dot{\Omega}_3^{-+}(\hat{N}) \\ &+ (-1)^{\hat{n}_1}\left(\dot{\Omega}_1^{+-}(\hat{M}) - \dot{\Omega}_3^{+-}(\hat{M}) + \dot{\Omega}_1^{++}(\hat{M}) - \dot{\Omega}_3^{++}(\hat{M})\right) \\ &+ \frac{1}{2}\ln(C_3)[1 - (-1)^{\hat{n}_1}] + \dot{v}(\hat{N})\Bigg]\Bigg\}, \end{aligned} \tag{32}$$

where $\dot{v}(\hat{N})$ is an arbitrary function of $\hat{N}$, and

$$\dot{\Omega}_k^{\pm\pm}(\hat{x}) \equiv \ln\left\{\Gamma\left[\frac{1}{4}\left(k \pm \hat{x} \pm \sqrt{-8C_1/C_3 - (\hat{N}^2 + 2\hat{N} - 1)}\right)\right]\right\} \tag{33}$$

in which the order of two superscripts $\pm\pm$ of $\dot{\Omega}$ is the same as that of them appearing in the equation of r.h.s., and $\Gamma[a(\hat{N})]$ is an operator function, whose expectation value in $\mathcal{F}$ in fact is the ordinary Gamma function $\Gamma[a(N)]$ for the real number $a(N)$, i.e.,

$$\langle n_1 n_2 | \, \Gamma[a(\hat{N})] \, | n_1 n_2 \rangle = \Gamma[a(N)]. \tag{34}$$

Different from the nonunitary realization (29), this nonunitary realization (30) may not be reduced to the Jordan–Schwinger realization (3) of SU(2) since in equations (32) and (33) C_3 can not take zero.

It will be verified later that the nonunitary realizations (29) and (30) may be connected with the unitary realization (26) by similarity transformations.

2. The (2, 2) case.

Setting $k = l = 2$ in equation (18) and taking $\alpha = 0$ into account, we may obtain two solutions. One of them is given by

$$\begin{aligned} \dot{f}_1^{(2,2)} \dot{g}_1^{(2,2)} = {} & [128(\hat{n}_1 - X_1^+(\hat{n}_1)/2)(\hat{n}_2 + 1)(\hat{n}_2 + 2)]^{-1} \\ & \times (\hat{n}_2 + X_3^+(\hat{n}_1)/2)\{16C_1 + C_3[\hat{n}_1^2 - X_1^-(\hat{n}_1)(\hat{n}_1 + 1) \\ & + \hat{n}_2(\hat{n}_2 + X_3^+(\hat{n}_1))]\}, \end{aligned} \tag{35}$$

where

$$X_k^{\pm}(\hat{n}_1) \equiv k \pm (-1)^{\hat{n}_1}. \tag{36}$$

Another solution may be directly get from equation (35) by considering the symmetry $\hat{n}_1 \leftrightarrow \hat{n}_2$. In the same way as discussing the $(1, 1)$ case, in terms of equation (35), we may obtain the unitary two-boson realization of quadratic type

$$\begin{aligned} \dot{B}_1^{(2,2)}(\mathcal{J}_+) = {} & [128(\hat{n}_1 - X_1^+(\hat{n}_1)/2)(\hat{n}_2 + 1)(\hat{n}_2 + 2)]^{-1/2} \\ & \times \{(\hat{n}_2 + X_3^+(\hat{n}_1)/2)\{16C_1 + C_3[\hat{n}_1^2 - X_1^-(\hat{n}_1)(\hat{n}_1 + 1) \\ & + \hat{n}_2(\hat{n}_2 + X_3^+(\hat{n}_1))]\}\}^{1/2}(a_1^+)^2 a_2^2, \\ \dot{B}_1^{(2,2)}(\mathcal{J}_-) = {} & a_1^2(a_2^+)^2[128(\hat{n}_1 - X_1^+(\hat{n}_1)/2)(\hat{n}_2 + 1)(\hat{n}_2 + 2)]^{-1/2} \\ & \times \{(\hat{n}_2 + X_3^+(\hat{n}_1)/2)\{16C_1 + C_3[\hat{n}_1^2 - X_1^-(\hat{n}_1)(\hat{n}_1 + 1) \\ & + \hat{n}_2(\hat{n}_2 + X_3^+(\hat{n}_1))]\}\}^{1/2}, \\ \dot{B}_1^{(2,2)}(\mathcal{J}_3) = {} & \tfrac{1}{4}(\hat{n}_1 - \hat{n}_2) \end{aligned} \tag{37}$$

and the nonunitary two-boson realization of quadratic type

$$\begin{aligned} \dot{B}_2^{(2,2)}(\mathcal{J}_+) = {} & [128(\hat{n}_1 - X_1^+(\hat{n}_1)/2)(\hat{n}_2 + 1)(\hat{n}_2 + 2)]^{-1} \\ & \times \{(\hat{n}_2 + X_3^+(\hat{n}_1)/2)\{16C_1 + C_3[\hat{n}_1^2 - X_1^-(\hat{n}_1)(\hat{n}_1 + 1) \\ & + \hat{n}_2(\hat{n}_2 + X_3^+(\hat{n}_1))]\}\}(a_1^+)^2 a_2^2, \\ \dot{B}_2^{(2,2)}(\mathcal{J}_-) = {} & a_1^2(a_2^+)^2, \\ \dot{B}_2^{(2,2)}(\mathcal{J}_3) = {} & \tfrac{1}{4}(\hat{n}_1 - \hat{n}_2). \end{aligned} \tag{38}$$

We observe that the unitary realization (37) is explicitly different from that of [18], in which $J_{\pm}$ and J_3 are first defined as $J_+ = (a_1^+)^k a_2^l$, $J_- = a_1^k (a_2^+)^l$ and $J_3 = (\hat{n}_1 - \hat{n}_2)/(k + l)$, however, in order to generate $\mathcal{H}$ the unique nontrivial choice is $k = l = 2$, combined with the coefficient of J_3^3, in the commutator $[J_+, J_-]$, being the fixed number -64, and the coefficient of J_3 in fact being the operator function of $\hat{N}$. However, the realization defined by equation (15) allows the arbitrary powers and the corresponding constant coefficients.

3.2. The second kind of realizations

In analogy with the Jordan–Schwinger realization (4) of SU(1,1), the second kind of two-boson realizations of $\mathcal{H}$ may be constructed in the following scheme:

$$\begin{aligned}\ddot{B}^{(k,l)}(\mathcal{J}_+) &= \ddot{f}(\hat{n}_1,\hat{n}_2)(a_1^+)^k(a_2^+)^l,\\ \ddot{B}^{(k,l)}(\mathcal{J}_-) &= a_1^k a_2^l\,\ddot{g}(\hat{n}_1,\hat{n}_2),\\ \ddot{B}^{(k,l)}(\mathcal{J}_3) &= \ddot{h}(\hat{n}_1,\hat{n}_2),\end{aligned}\tag{39}$$

where k and l are positive integers, the operator functions $\ddot{f}(\hat{n}_1,\hat{n}_2)$, $\ddot{g}(\hat{n}_1,\hat{n}_2)$ and $\ddot{h}(\hat{n}_1,\hat{n}_2)$ have to be determined by the commutation relations (11) of $\mathcal{H}$. Acting $\ddot{B}^{(k,l)}(\mathcal{J}_\pm)$ for a fixed (k,l) on some basis vector $|n_1n_2\rangle$ of $\mathcal{F}$ produces another basis vector $|n_1 \pm k, n_2 \pm l\rangle$.

It follows that inserting equation (39) into the first equation of equation (11) leads to the difference equation

$$\ddot{h}(\hat{n}_1,\hat{n}_2) - \ddot{h}(\hat{n}_1 - k, \hat{n}_2 - l) = 1.\tag{40}$$

Its solution reads

$$\ddot{h}(\hat{n}_1,\hat{n}_2) = \frac{\hat{n}_1}{2k} + \frac{\hat{n}_2}{2l} + \beta,\tag{41}$$

where the real constant β will be determined later.

Using equation (41), in order to satisfy the second equation of equation (11), the following difference equation must hold:

$$\begin{aligned}&\left[\prod_{i=1}^{k}(\hat{n}_1 - i + 1)\right]\left[\prod_{i=1}^{l}(\hat{n}_2 - i + 1)\right]\ddot{f}^{(k,l)}(\hat{n}_1,\hat{n}_2)\ddot{g}^{(k,l)}(\hat{n}_1,\hat{n}_2)\\ &-\left[\prod_{i=1}^{k}(\hat{n}_1 + i)\right]\left[\prod_{i=1}^{l}(\hat{n}_2 + i)\right]\ddot{f}^{(k,l)}(\hat{n}_1 + k,\hat{n}_2 + l)\ddot{g}^{(k,l)}(\hat{n}_1 + k,\hat{n}_2 + l)\\ &= C_1\left(\frac{\hat{n}_1}{2k} + \frac{\hat{n}_2}{2l} + \beta\right) + C_3\left(\frac{\hat{n}_1}{2k} + \frac{\hat{n}_2}{2l} + \beta\right)^3.\end{aligned}\tag{42}$$

Just like the first kind of realizations discussed in the last subsection, in what follows, we will study the case of $(k,l) = (1,1)$, and give directly the results of $(k,l) = (2,2)$.

1. The (1,1) case.

Solving equation (42) with setting $k = l = 1$, we have two solutions:

$$\begin{aligned}\ddot{f}_1^{(1,1)}(\hat{n}_1,\hat{n}_2)\ddot{g}_1^{(1,1)}(\hat{n}_1,\hat{n}_2) = -\frac{1}{8\hat{n}_1}(\hat{n}_1 + 2\beta - 1)\{4C_1 + C_3[\hat{n}_1(\hat{n}_1 + 4\beta - 2)\\ +\hat{n}_2^2 + 4\beta(\beta - 1)]\},\end{aligned}\tag{43}$$

and

$$\ddot{f}_2^{(1,1)}(\hat{n}_1,\hat{n}_2)\ddot{g}_2^{(1,1)}(\hat{n}_1,\hat{n}_2) = -\frac{1}{8\hat{n}_2}(\hat{n}_2+2\beta-1)\{4C_1+C_3[\hat{n}_1^2+\hat{n}_2(\hat{n}_2+4\beta-2) +4\beta(\beta-1)]\}. \quad (44)$$

Between the two solutions there exists explicitly the symmetry: $\hat{n}_1 \leftrightarrow \hat{n}_2$, so we need merely to consider the solution (43).

(1) If the unitary relations $\ddot{B}^{(1,1)}(\mathcal{J}_\pm) = (\ddot{B}^{(1,1)}(\mathcal{J}_\mp))^\dagger$ are imposed, namely, $\ddot{f}_1^{(1,1)}(\hat{n}_1,\hat{n}_2) = \ddot{g}_1^{(1,1)}(\hat{n}_1,\hat{n}_2)$, then solving equation (43) and substituting it into equation (39), we obtain

$$\begin{aligned}
\ddot{B}_1^{(1,1)}(\mathcal{J}_+) &= \{-\tfrac{1}{8\hat{n}_1}(\hat{n}_1+2\beta-1)\{4C_1+C_3[\hat{n}_1(\hat{n}_1+4\beta-2) \\
&\quad +\hat{n}_2^2+4\beta(\beta-1)]\}\}^{1/2}a_1^+a_2^+, \\
\ddot{B}_1^{(1,1)}(\mathcal{J}_-) &= a_1a_2\{-\tfrac{1}{8\hat{n}_1}(\hat{n}_1+2\beta-1)\{4C_1+C_3[\hat{n}_1(\hat{n}_1+4\beta-2) \\
&\quad +\hat{n}_2^2+4\beta(\beta-1)]\}\}^{1/2}, \\
\ddot{B}_1^{(1,1)}(\mathcal{J}_3) &= \tfrac{1}{2}(\hat{n}_1+\hat{n}_2)+\beta.
\end{aligned} \quad (45)$$

Substituting equation (45) into equation (12), the Casimir invariant $\mathcal{C}$ of $\mathcal{H}$ may be expressed in terms of $\hat{n}_1$ and $\hat{n}_2$ as

$$\mathcal{C} = \tfrac{1}{64}(\hat{M}+2\beta-2)(\hat{M}+2\beta)[8C_1+C_3(\hat{M}+2\beta-2)(\hat{M}+2\beta)], \quad (46)$$

where $\hat{M} = \hat{n}_1-\hat{n}_2$ or $\hat{n}_2-\hat{n}_1$ is the number *difference* operator for two kinds of different bosons, while in equation (24), the boson number *sum* operator, i.e., the total boson number operator $\hat{N}$, appears. Calculating $\langle n_1n_2|\mathcal{C}|n_1n_2\rangle$ and then comparing it with the forth equation of equation (15) gives

$$\tilde{j} = \tfrac{1}{2}(M+2\beta-2) \text{ or } \tilde{j} = \tfrac{1}{2}(M-2\beta), \quad (47)$$

where $M = n_1-n_2$ or n_2-n_1 is the eigenvalue of $\hat{M}$. The symmetry requires that $\beta = 1/2$, thus, the irreducible representation $\tilde{j}$ of $\mathcal{H}$ are related to M through the equation $\tilde{j} = \frac{1}{2}(M-1)$. SU(1,1) has the similar result [5]. Correspondingly, equation (45) becomes

$$\begin{aligned}
\ddot{B}_2^{(1,1)}(\mathcal{J}_+) &= \sqrt{-\tfrac{1}{2}C_1-\tfrac{1}{8}C_3(\hat{n}_1^2+\hat{n}_2^2-1)}\,a_1^+a_2^+, \\
\ddot{B}_2^{(1,1)}(\mathcal{J}_-) &= a_1a_2\sqrt{-\tfrac{1}{2}C_1-\tfrac{1}{8}C_3(\hat{n}_1^2+\hat{n}_2^2-1)}, \\
\ddot{B}_2^{(1,1)}(\mathcal{J}_3) &= \tfrac{1}{2}(\hat{n}_1+\hat{n}_2+1).
\end{aligned} \quad (48)$$

Thus, for $C_3 \neq 0$ the spaces that the operators $\ddot{B}_2^{(1,1)}(\mathcal{J}_\mu)$ ($\mu = \pm$, 3) act on may be certain subspaces of the Fock space $\mathcal{F} = \{|n_1n_2\rangle \,|\, n_1, n_2 = 0,1,2,\ldots\}$, that is, n_1 and n_2 need limiting in order that the values of the square roots appeared in the matrix elements $\langle n_1 \pm 1 n_2 \pm 1|\ddot{B}_2^{(1,1)}(\mathcal{J}_\pm)|n_1 n_2\rangle$ must be greater than or equal to zero. For the realization (48), n_1 and n_2 have to satisfy the constraint equation

$$n_1^2 + n_2^2 \geqslant 1 - 4\frac{C_1}{C_3}, \tag{49}$$

whose results are listed as follows.

(A) If $C_1 \geqslant C_3/4$, then equation (49) always holds, so that the acting space of $\ddot{B}_2^{(1,1)}(\mathcal{J}_\mu)$ is the whole Fock space $\mathcal{F}$, in which

$$\{|0\, n_2\rangle \,|\, n_2 = 0,\ 1,\ 2, \ldots\} \quad \text{and} \quad \{|n_1\, 0\rangle \,|\, n_1 = 0,\ 1,\ 2, \ldots\}$$

are the infinite-dimensional nullspaces of $\ddot{B}_2^{(1,1)}(\mathcal{J}_-)$, since they satisfy

$$\ddot{B}_2^{(1,1)}(\mathcal{J}_-)|0\, n_2\rangle = \ddot{B}_2^{(1,1)}(\mathcal{J}_-)|n_1\, 0\rangle = 0.$$

(B) If $C_1 < C_3/4$, then the values of n_1 and n_2 need limiting. First consider that n_1 takes independently values, then the values that n_2 may take are dependent on n_1, especially, its smallest values should be $\kappa_1(n_1) \equiv \left[\sqrt{1 - 4C_1/C_3 - n_1^2}\right]$ for the given n_1. As a result, the acting subspace of $\ddot{B}_2^{(1,1)}(\mathcal{J}_\mu)$ is

$$\ddot{V}_1 = \bigcup_{n_1=0}^{\lambda} \ddot{V}_1(n_1),$$

where

$$\ddot{V}_1(n_1) \equiv \{|n_1, \kappa_1(n_1) + i\rangle \,|\, i = 0,\ 1, \ldots\}, \qquad \lambda \equiv \left[\sqrt{1 - 4C_1/C_3}\right] - 1.$$

In $\ddot{V}_1$, $\ddot{B}_2^{(1,1)}(\mathcal{J}_-)$ has an infinite-dimensional nullspace $\ddot{V}_1(0)$ and a λ-dimensional nullspace $\{|n_1, \kappa_1(n_1)\rangle \,|\, n_1 = 1, 2, \ldots, \lambda\}$.

Secondly, n_2 takes independently values, by means of the symmetry $n_1 \leftrightarrow n_2$ of equation (49), then the smallest value of n_1 is $\kappa_2(n_2) \equiv \left[\sqrt{1 - 4C_1/C_3 - n_2^2}\right]$, so that the acting space of $\ddot{B}_2^{(1,1)}(\mathcal{J}_\mu)$ is

$$\ddot{V}_2 = \bigcup_{n_2=0}^{\lambda} \ddot{V}_2(n_2) \equiv \bigcup_{n_2=0}^{\lambda} \{|\kappa_2(n_2) + i, n_2\rangle \,|\, i = 0,\ 1, \ldots\}.$$

Obviously, in $\ddot{V}_2$, $\ddot{V}_2(0)$ and $\{|\kappa_2(n_2), n_2\rangle \,|\, n_2 = 1, 2, \ldots, \lambda\}$ are the nullspaces of $\ddot{B}_2^{(1,1)}(\mathcal{J}_-)$ with infinite-dimension and λ-dimension, respectively.

However, for the second kind of realization (48), $\ddot{B}_2^{(1,1)}(\mathcal{J}_+)$ and $\ddot{B}_2^{(1,1)}(\mathcal{J}_-)$ have no the common nullspace state.

(2) If the unitary relations need not satisfying, it follows from equation (45) that the conventional choice $\ddot{g}^{(1,1)}(\hat{n}_1, \hat{n}_2) = 1$ (or $\ddot{f}^{(1,1)}(\hat{n}_1, \hat{n}_2) = 1$) results in the following nonunitary two-boson realization

$$\begin{aligned} \ddot{B}_3^{(1,1)}(\mathcal{J}_+) &= -\tfrac{1}{8\hat{n}_1}(\hat{n}_1 + 2\beta - 1)\{4C_1 + C_3[\hat{n}_1(\hat{n}_1 + 4\beta - 2) \\ &\quad + \hat{n}_2^2 + 4\beta(\beta - 1)]\}a_1^+ a_2^+, \\ \ddot{B}_3^{(1,1)}(\mathcal{J}_-) &= a_1 a_2, \\ \ddot{B}_3^{(1,1)}(\mathcal{J}_3) &= \tfrac{1}{2}(\hat{n}_1 + \hat{n}_2) + \beta. \end{aligned} \tag{50}$$

Taking $\beta = 1/2$, equation (50) becomes

$$\begin{aligned} \ddot{B}_4^{(1,1)}(\mathcal{J}_+) &= -\left[\tfrac{1}{2}C_1 + \tfrac{1}{8}C_3(\hat{n}_1^2 + \hat{n}_2^2 - 1)\right] a_1^+ a_2^+, \\ \ddot{B}_4^{(1,1)}(\mathcal{J}_-) &= a_1 a_2, \\ \ddot{B}_4^{(1,1)}(\mathcal{J}_3) &= \tfrac{1}{2}(\hat{n}_1 + \hat{n}_2 + 1). \end{aligned} \tag{51}$$

When $C_1 = -2$ and $C_3 = 0$, equation (51), together with equation (48), recovers the unitary Jordan–Schwinger realization (4) of SU(1,1).

(3) Choosing $\ddot{g}(\hat{n}_1, \hat{n}_2) = \ddot{f}(\hat{n}_1 - 1, \hat{n}_2 - 1)$ and $\beta = 1/2$ in equation (44), we may obtain another constrained nonunitary realization

$$\begin{aligned} \ddot{B}_5^{(1,1)}(\mathcal{J}_+) &= \ddot{f}(\hat{n}_1, \hat{n}_2) a_1^+ a_2^+, \\ \ddot{B}_5^{(1,1)}(\mathcal{J}_-) &= \ddot{f}(\hat{n}_1, \hat{n}_2) a_1 a_2, \\ \ddot{B}_5^{(1,1)}(\mathcal{J}_3) &= \tfrac{1}{2}(\hat{n}_1 + \hat{n}_2 + 1), \end{aligned} \tag{52}$$

where $\ddot{f}(\hat{n}_1, \hat{n}_2)$ obeys

$$8\ddot{f}(\hat{n}_1, \hat{n}_2) = -\left[4C_1 + C_3(\hat{n}_1^2 + \hat{n}_2^2 - 1)\right] \ddot{f}^{-1}(\hat{n}_1 - 1, \hat{n}_2 - 1), \tag{53}$$

whose solution is

$$\begin{aligned} \ddot{f}(\hat{n}_1, \hat{n}_2) = \exp\Big\{(-1)^{\hat{n}_1 - 1}\Big[&-\ddot{\Omega}_2^{--}(\hat{M}) + \ddot{\Omega}_4^{--}(\hat{M}) - \ddot{\Omega}_2^{-+}(\hat{M}) + \ddot{\Omega}_4^{-+}(\hat{M}) \\ &+ (-1)^{\hat{n}_1}\Big(\ddot{\Omega}_2^{+-}(\hat{N}) - \ddot{\Omega}_4^{+-}(\hat{N}) + \ddot{\Omega}_2^{++}(\hat{N}) - \ddot{\Omega}_4^{++}(\hat{N})\Big) \\ &+ \frac{1}{2}(\ln(C_3) + \mathrm{i}\pi)[1 - (-1)^{\hat{n}_1}] + \ddot{v}(\hat{M})\Big]\Big\}, \end{aligned} \tag{54}$$

where $\ddot{v}(\hat{M})$ is an arbitrary function of $\hat{M}$, and

$$\ddot{\Omega}_k^{\pm\pm}(\hat{x}) \equiv \ln\left\{\Gamma\left[\frac{1}{4}\left(k \pm \hat{x} \pm \sqrt{-8C_1/C_3 - (\hat{M}^2 - 2)}\right)\right]\right\}, \tag{55}$$

in which $\Gamma[a(\hat{N})]$ has been defined by equation (34). The nonunitary realization (52) can not become the Jordan–Schwinger realization (4) of SU(1,1) on account of the singularity of C_3 in equations (54) and (55).

We notice that all the nonunitary realizations, (29), (30), (51) and (52), obtained above are different from the inhomogeneous two-boson realizations obtained in [24] by using the boson mapping method based upon the induced representations of $\mathcal{H}$ on the quotient spaces $U(\mathcal{H})/I_i$ ($i = 1, 2$), where $U(\mathcal{H})$ is the universal enveloping algebra of $\mathcal{H}$ and I_i are two left ideals with respect to $U(\mathcal{H})$.

2. The (2, 2) case.

Equation (42) with setting $k = l = 2$ and $\beta = 1/2$ has two solutions, the first one is given by

$$\begin{aligned}\ddot{f}_1^{(2,2)}\ddot{g}_1^{(2,2)} = {} & [128(\hat{n}_1 - 1)\hat{n}_1(\hat{n}_2 - 1)n_2]^{-1}(\hat{n}_1 + X_1^-(\hat{n}_1)/2) \\ & \times(\hat{n}_2 + 2 - X_5^-(\hat{n}_1)/2)\{16C_1 + C_3[X_1^-(\hat{n}_1)(\hat{n}_1 + 3) + \hat{n}_1^2 \\ & + \hat{n}_2(\hat{n}_2 + 4 + X_3^+(\hat{n}_1)) + 2(2 - X_3^+(\hat{n}_1))]\},\end{aligned} \tag{56}$$

where the symbol $X_k^{\pm}(\hat{n}_1)$ has be defined by equation (36). The second solution may be directly obtained from equation (56) by the substitutions $\hat{n}_1 \to \hat{n}_2$ and $\hat{n}_2 \to \hat{n}_1$. Solving equation (56) by considering respectively the unitary and nonunitary conditions, and then inserting them into equation (39), we may obtain for $\mathcal{H}$ the unitary two-boson realization of quadratic type

$$\begin{aligned}\ddot{B}_1^{(2,2)}(\mathcal{J}_+) = {} & [128(\hat{n}_1 - 1)\hat{n}_1(\hat{n}_2 - 1)n_2]^{-1/2}[\hat{n}_1 + X_1^-(\hat{n}_1)/2] \\ & \times[\hat{n}_2 + 2 - X_5^-(\hat{n}_1)/2]\{16C_1 + C_3[X_1^-(\hat{n}_1)(\hat{n}_1 + 3) + \hat{n}_1^2 \\ & + \hat{n}_2(\hat{n}_2 + 4 + X_3^+(\hat{n}_1)) + 2(2 - X_3^+(\hat{n}_1))]\}^{1/2}(a_1^+)^2(a_2^+)^2, \\ \dot{B}_1^{(2,2)}(\mathcal{J}_-) = {} & a_1^2 a_2^2[128(\hat{n}_1 - 1)\hat{n}_1(\hat{n}_2 - 1)n_2]^{-1/2}[\hat{n}_1 + X_1^-(\hat{n}_1)/2] \\ & \times[\hat{n}_2 + 2 - X_5^-(\hat{n}_1)/2]\{16C_1 + C_3[X_1^-(\hat{n}_1)(\hat{n}_1 + 3) + \hat{n}_1^2 \\ & + \hat{n}_2(\hat{n}_2 + 4 + X_3^+(\hat{n}_1)) + 2(2 - X_3^+(\hat{n}_1))]\}^{1/2}, \\ \dot{B}_1^{(2,2)}(\mathcal{J}_3) = {} & \tfrac{1}{4}(\hat{n}_1 + \hat{n}_2 + 2)\end{aligned} \tag{57}$$

and the nonunitary two-boson realization of quadratic type

$$\begin{aligned}\ddot{B}_2^{(2,2)}(\mathcal{J}_+) = {} & [128(\hat{n}_1 - 1)\hat{n}_1(\hat{n}_2 - 1)n_2]^{-1}[\hat{n}_1 + X_1^-(\hat{n}_1)/2] \\ & \times[\hat{n}_2 + 2 - X_5^-(\hat{n}_1)/2]\{16C_1 + C_3[X_1^-(\hat{n}_1)(\hat{n}_1 + 3) + \hat{n}_1^2 \\ & + \hat{n}_2(\hat{n}_2 + 4 + X_3^+(\hat{n}_1)) + 2(2 - X_3^+(\hat{n}_1))]\}(a_1^+)^2(a_2^+)^2, \\ \dot{B}_2^{(2,2)}(\mathcal{J}_-) = {} & a_1^2 a_2^2, \\ \dot{B}_2^{(2,2)}(\mathcal{J}_3) = {} & \tfrac{1}{4}(\hat{n}_1 + \hat{n}_2 + 2).\end{aligned} \tag{58}$$

4. Unitarization equations and similarity transformations

In the last section, the two kinds of two-boson realizations of $\mathcal{H}$ are constructed, and in each kind, one unitary realization and two different nonunitary realizations are discussed, respectively. In this section, we will show that the unitary realizations and the nonunitary realizations in the same kind may be connected by similarity transformations.

Let us begin with discussing the general procedure. Denote the unitary boson realization and the nonunitary boson realization by $B^{\mathrm{u}}(\mathcal{J}_\mu)$ ($\mu = \pm, 3$)

and $B^{\mathrm{nu}}(\mathcal{J}_\mu)$, respectively, and the corresponding similarity transformation by S, then we have

$$SB^{\mathrm{nu}}(\mathcal{J}_\mu)S^{-1} = B^{\mathrm{u}}(\mathcal{J}_\mu). \tag{59}$$

Hence, S in general is an operator function with respect to the boson operators and the particle number operators.

Using equation (59) and the unitary conditions satisfied by $B^{\mathrm{u}}(\mathcal{J}_\mu)$

$$\begin{aligned}(B^{\mathrm{u}}(\mathcal{J}_\pm))^\dagger &= B^{\mathrm{u}}(\mathcal{J}_\mp),\\ (B^{\mathrm{u}}(\mathcal{J}_3))^\dagger &= B^{\mathrm{u}}(\mathcal{J}_3),\end{aligned} \tag{60}$$

it follows that we may obtain the following unitarization equations obeyed by $B^{\mathrm{nu}}(\mathcal{J}_\mu)$

$$\begin{aligned}U^{-1}\,(B^{\mathrm{nu}}(\mathcal{J}_\pm))^\dagger\,U &= B^{\mathrm{nu}}(\mathcal{J}_\mp),\\ U^{-1}\,(B^{\mathrm{nu}}(\mathcal{J}_3))^\dagger\,U &= B^{\mathrm{nu}}(\mathcal{J}_3),\end{aligned} \tag{61}$$

where $U \equiv S^\dagger S$ is an Hermitian operator. The similarity transformation S may be obtained by solving equation (61) in the Fock space.

We observe from the two-boson realizations (29), (30), (51) and (52) obtained in the last section that $\dot{B}_4^{(1,1)}(\mathcal{J}_3)$, $\dot{B}_5^{(1,1)}(\mathcal{J}_3)$, $\ddot{B}_4^{(1,1)}(\mathcal{J}_3)$ and $\ddot{B}_5^{(1,1)}(\mathcal{J}_3)$ in fact are already Hermitian, so equation (59) implies that the corresponding similarity transformations commute with $\mathcal{J}_3$, in other words, they depend only on the particle number operators, $\hat{n}_1$ and $\hat{n}_2$.

Now let us seek the similarity transformations S_1 and S_2 that correspond to the nonunitary realizations (29) and (51), respectively. Calculating the matrix elements of the unitarization equations (see equation (61)) satisfied respectively by $\dot{B}_4^{(1,1)}(\mathcal{J}_-)$ and $\ddot{B}_4^{(1,1)}(\mathcal{J}_-)$ in the Fock space $\mathcal{F}$, and using equations (29) and (51), we may deduce the recurrent equations satisfied by the expectation values $S_i(n_1, n_2) \equiv \langle n_1 n_2 | S_i | n_1 n_2 \rangle$ $(i = 1, 2)$,

$$\left\{4C_1 + C_3[n_1^2 + n_2(n_2+2)]\right\} S_1(n_1, n_2)^2 = 8S_1(n_1 - 1, n_2 + 1)^2, \tag{62}$$

and

$$\left[4C_1 + C_3(n_1^2 + n_2^2 - 1)\right] S_2(n_1, n_2)^2 = -8S_2(n_1 - 1, n_2 - 1)^2. \tag{63}$$

Solving equations (62) and (63), and then using equation (7), we obtain

$$S_1(\hat{n}_1, \hat{n}_2) = \sqrt{\frac{(C_3/4)^{1-\hat{n}_1}\,\dot{w}(\hat{N})}{(\dot{Z}(\hat{N})_+)_{\hat{n}_1-1}(\dot{Z}(\hat{N})_-)_{\hat{n}_1-1}}}, \tag{64}$$

and

$$S_2(\hat{n}_1, \hat{n}_2) = \sqrt{-\frac{(-1)^{\hat{n}_1}(C_3/4)^{1-\hat{n}_1}\,\ddot{w}(\hat{M})}{(\ddot{Z}(\hat{M})_+)_{\hat{n}_1-1}(\ddot{Z}(\hat{M})_-)_{\hat{n}_1-1}}}, \tag{65}$$

respectively. In the above two equations, the minus signs out of the square-root symbols have been omitted without loss of generality, $\dot{w}(\hat{N})$ and $\ddot{w}(\hat{M})$ are arbitrary functions with respect to the sum operator $\hat{N}$ and the difference operator $\hat{M}$, respectively,

$$\dot{Z}(\hat{N})_{\pm} \equiv \frac{1}{2}\left[3 - \hat{N} \pm \sqrt{-8C_1/C_3 - (\hat{N}^2 + 2\hat{N} - 1)}\right], \tag{66}$$

$$\ddot{Z}(\hat{M})_{\pm} \equiv \frac{1}{2}\left[4 - \hat{M} \pm \sqrt{-8C_1/C_3 - (\hat{M}^2 - 2)}\right], \tag{67}$$

and the symbol $(Z(\hat{N}))_{\hat{n}_i}$ in denominators stands for an operator function of $\hat{N}$ and $\hat{n}_i$, whose expectation value in $\mathcal{F}$ is the usual Pochhammer symbol $(Z(N))_{n_i}$ for the real number $Z(N)$ and the positive integer n_i, i.e.,

$$\langle n_1 n_2 |\, (Z(\hat{N}))_{\hat{n}_i}\, | n_1 n_2 \rangle = Z(N)[Z(N)+1]\ldots[Z(N)+n_i-1] \equiv (Z(N))_{n_i}. \tag{68}$$

For the constrained nonunitary realizations (30) and (52), there must exist the corresponding similarity transformations $\bar{S}_1$ and $\bar{S}_2$, which connect equation (30) with equation (26), and equation (52) with equation (48), respectively. Using the same calculating method, we may obtain

$$\bar{S}_1(\hat{n}_1, \hat{n}_2) = \sqrt{\frac{8}{4C_1 + C_3[\hat{n}_1^2 + \hat{n}_2(\hat{n}_2 + 2)]}}, \tag{69}$$

and

$$\bar{S}_2(\hat{n}_1, \hat{n}_2) = \sqrt{-\frac{8}{4C_1 + C_3(\hat{n}_1^2 + \hat{n}_2^2 - 1)}}. \tag{70}$$

5. Some applications

In this section, as applications, we shall apply the results obtained previously to discussing the dynamical symmetry of the Kepler system in the two-dimensional curved space and to constructing phase operators of $\mathcal{H}$.

5.1. Dynamical symmetry of the Kepler system in the two-dimensional curved space

The key idea of dynamical symmetry is that the Hamiltonian describing some quantum system can be constructed in terms of the Casimir invariants, $C(g_1)$, $C(g_2), \ldots,$ of a chain of algebras $g_1 \supset g_2 \supset \ldots.$ [25]. The most famous example of the dynamical symmetry is the nonrelativistic hydrogen atom, [26] whose Hamiltonian H^{c} can be expressed by the first quadratic Casimir invariant, $C(\mathrm{SO}(4))$, of the SO(4) algebra, which is spanned by the three components of the angular momentum **J** and the three components of the Runge–Lentz–Laplace vector **R**, as $H^{\mathrm{c}} \sim [C(\mathrm{SO}(4))+1]^{-1}$. As mentioned in Section 1, Higgs has showed that the Kepler system in the two-dimensional curved space is governed by the Higgs algebra $\mathcal{H}$, and however, he applied the SO(3) algebra to calculate its energy levels. In this subsection, we will show that the Hamiltonian H of this Kepler system may be naturally related to the Casimir invariant $\mathcal{C}$ of $\mathcal{H}$, and then obtain directly the energy levels of H by using the eigenvalue of $\mathcal{C}$.

The Hamiltonian of the Kepler system in the two-dimensional curved space has the following expression [8]

$$H = \frac{1}{2}\left(\pi_i \pi_i + \lambda \mathcal{J}_3^2\right) - \frac{\mu}{r}, \tag{71}$$

where λ is the curvature of the sphere, μ is a constant number, $\mathcal{J}_3$ is a two-dimensional rotation operator, and π_i ($i = 1, 2$), the two components of the momentum operator $\vec{\pi}$ in the two-dimensional curved space, are defined by

$$\pi_i = p_i - \frac{\lambda}{2}\left\{x_i,\ (\mathbf{x}\cdot\mathbf{p})\right\}, \tag{72}$$

where $\{\ ,\ \}$ is the usual anticommutator, $p_i = -\partial_{x_i}$ ($i = 1, 2$) are the two components of the ordinary momentum operator **p** conjugate to **x**, respectively.

This system possess three constants of motion: one is $\mathcal{J}_3$, the remaining two are the two components of the Runge–Lentz–Laplace vector **R** in the two-dimensional curved space, which, in analogy with those in the three-dimensional flat space, [5,26] may be constructed as

$$R_i = \frac{1}{2}\{\mathcal{J}_3,\ \epsilon_{ij}\pi_j\} + \mu\frac{x_i}{r}, \quad i = 1,\ 2, \tag{73}$$

where ϵ_{ij} is the two-dimensional Levi–Civita symbol.

It can be easily verified that $\mathcal{J}_3$ and $R_\pm = R_1 \pm \mathrm{i}R_2$ satisfy

$$\begin{aligned}
&[\mathcal{J}_3, R_\pm] = \pm R_\pm,\\
&[R_+, R_-] = \left(\tfrac{\lambda}{2} - 4H\right)\mathcal{J}_3 + 4\lambda\mathcal{J}_3^3,
\end{aligned} \tag{74}$$

and

$$\{R_+,\ R_-\} = 2\mu^2 + \left(2H - \lambda\mathcal{J}_3^2\right)\left(2\mathcal{J}_3^2 + \tfrac{1}{2}\right) - 2\lambda\mathcal{J}_3^2. \tag{75}$$

If the state vector space on which equation (74) is allowed to act is the energy eigenspace, then the Hamiltonian H in equation (74) may be replaced by the corresponding energy eigenvalue E, as the result, equation (74) can be put in the form of the Higgs algebra, equation (11), with

$$C_1 = \tfrac{1}{2}\lambda - 4E, \quad C_3 = 4\lambda. \tag{76}$$

Using equations (12), (74) and (75), as expected, there indeed exists a simple relation between H and the Casimir invariant $\mathcal{C}$ of $\mathcal{H}$, i.e.

$$H = 2(\mathcal{C} - \mu^2). \tag{77}$$

It follows that calculation of the expectation value of equation (77) in the Fock space $\mathcal{F}$, with the help of equation (24) with setting $\alpha = 0$ and equation (76), leads immediately to the following equation satisfied by E

$$E = -2\mu^2 + \left(-E + \tfrac{1}{8}\lambda\right) N(N+2) + \tfrac{1}{8}\lambda N^2(N+2)^2, \tag{78}$$

whose solution reads

$$E_N = \frac{\lambda}{8} N(N+2) - \frac{2\mu^2}{(N+1)^2}. \tag{79}$$

This result may also be obtained by using the Casimir invariant (46) of $\mathcal{H}$ in the second kind of two-boson realizations. Owing to the fact that E_N depends only to N rather than n_1 and n_2, the degeneracy of the energy level for the fixed N is $N + 1$. The physical condition that the quantum number $\tilde{m}(= \frac{1}{2}(n_1 - n_2))$ of $\mathcal{J}_3$ must be the nonnegative integers requires that $N(= \frac{1}{2}(n_1 + n_2))$ has to take the nonnegative even numbers, i.e., 0, 2, 4, … . If let $N = 2n$ $(n = 0, 1, 2, \ldots)$, then equation (79) becomes the result (53) of [8]. If the two parameters λ and μ in equation (79) satisfy the following condition

$$\frac{\mu^2}{\lambda} = l \left(l + \tfrac{1}{2}\right)^2 (l+1), \tag{80}$$

where l is some positive integer, then a zero energy level appears at $N = 2l$, i.e., $E_{2l} = 0$, while there exist l bounded states, $E_{2i} < 0$ $(i = 0, 1, \ldots, l-1)$, and infinite scattering states, $E_{2j} > 0$ $(j = l+1, l+2, \ldots)$.

5.2. *Phase operators of $\mathcal{H}$*

It is well known that the photon phase operators, introduced originally by Dirac [27] and amended by Susskind and Glogower [28], may be defined in terms of one set of boson operator $\{a_1^+, a_1, \hat{n}_1\}$ as [29]

$$\begin{aligned} \exp(\mathrm{i}\phi_1) &= \frac{1}{\sqrt{\hat{n}_1+1}} a_1, \\ \exp(-\mathrm{i}\phi_1) &= a_1^+ \frac{1}{\sqrt{\hat{n}_1+1}} = (\exp(\mathrm{i}\phi_1))^\dagger. \end{aligned} \tag{81}$$

It is easily shown that the above two operators satisfy

$$\begin{aligned} \exp(\mathrm{i}\phi_1)|n_1\rangle &= (1-\delta_{n_1 0})|n_1-1\rangle, \\ \exp(-\mathrm{i}\phi_1)|n_1\rangle &= |n_1+1\rangle, \end{aligned} \tag{82}$$

and

$$\begin{aligned} (\exp(-\mathrm{i}\phi_1))^\dagger \exp(-\mathrm{i}\phi_1) &= 1, \\ \exp(-\mathrm{i}\phi_1)(\exp(-\mathrm{i}\phi_1))^\dagger &= 1-|0\rangle\langle 0|, \end{aligned} \tag{83}$$

hence, we call $\exp(\pm\mathrm{i}\phi_1)$ semiunitary operators. If introduce the following two Hermitian phase operators

$$\begin{aligned} \cos\phi_1 &= \tfrac{1}{2}[\exp(\mathrm{i}\phi_1)+\exp(-\mathrm{i}\phi_1)], \\ \sin\phi_1 &= \tfrac{1}{2\mathrm{i}}[\exp(\mathrm{i}\phi_1)-\exp(-\mathrm{i}\phi_1)], \end{aligned} \tag{84}$$

then, they, together with $\hat{n}_1$, satisfy

$$\begin{aligned} [\hat{n}_1,\ \cos\phi_1] &= -\mathrm{i}\sin\phi_1, \\ [\hat{n}_1,\ \sin\phi_1] &= \mathrm{i}\cos\phi_1. \end{aligned} \tag{85}$$

For the Higgs algebra $\mathcal{H}$, making use of the first kind of two-boson realization, equation (26), we may construct the following two operators

$$\begin{aligned} \mathcal{E}_+ &= \frac{2}{\sqrt{\hat{n}_1+1}}\dot{B}_2^{(1,1)}(\mathcal{J}_-)\frac{1}{\sqrt{(\hat{n}_2+1)[2C_1+C_3\hat{n}_1(\hat{n}_2+1)]}}, \\ \mathcal{E}_- &= \frac{2}{\sqrt{(\hat{n}_2+1)[2C_1+C_3\hat{n}_1(\hat{n}_2+1)]}}\dot{B}_2^{(1,1)}(\mathcal{J}_+)\frac{1}{\sqrt{\hat{n}_1+1}} = (\mathcal{E}_+)^\dagger. \end{aligned} \tag{86}$$

We call $\mathcal{E}_\pm$ the phase operators of $\mathcal{H}$, since action of $\mathcal{E}_\pm$ on the eigenvector $|\tilde{j}\tilde{m}\rangle$, using equations (9), (10), (14) and (26), leads to

$$\begin{aligned} \mathcal{E}_+|\tilde{j}\tilde{m}\rangle &= (1-\delta_{-\tilde{j}\tilde{m}})|\tilde{j}\tilde{m}-1\rangle, \\ \mathcal{E}_-|\tilde{j}\tilde{m}\rangle &= (1-\delta_{\tilde{j}\tilde{m}})|\tilde{j}\tilde{m}+1\rangle. \end{aligned} \tag{87}$$

When C_1=2 and C_3=0, equation (86) becomes the phase operators of the angular momentum system [30]. Note that equation (87) is in fact the same as the equation satisfied by the phase operators of the angular momentum system.

Using equation (81), equation (86) can also be written in the following form

$$\begin{aligned} \mathcal{E}_+ &= \exp[\mathrm{i}(\phi_1-\phi_2)]w(\hat{n}_1,\hat{n}_2), \\ \mathcal{E}_- &= w(\hat{n}_1,\hat{n}_2)\exp[-\mathrm{i}(\phi_1-\phi_2)], \end{aligned} \tag{88}$$

where $\exp[\pm\mathrm{i}(\phi_1-\phi_2)]$ are the ordinary phase difference operators of two-dimensional harmonic oscillator, i.e.,

$$\begin{aligned} \exp[\mathrm{i}(\phi_1-\phi_2)] &= \frac{1}{\sqrt{\hat{n}_1+1}}a_1a_2^+\frac{1}{\sqrt{\hat{n}_2+1}}, \\ \exp[-\mathrm{i}(\phi_1-\phi_2)] &= \frac{1}{\sqrt{\hat{n}_2+1}}a_2a_1^+\frac{1}{\sqrt{\hat{n}_1+1}}, \end{aligned} \tag{89}$$

and the operator function $w(\hat{n}_1, \hat{n}_2)$ is given by

$$w(\hat{n}_1, \hat{n}_2) = \frac{2\dot{f}_1^{(1,1)}(\hat{n}_1, \hat{n}_2)}{\sqrt{2C_1 + C_3\hat{n}_1(\hat{n}_2 + 1)}} = \sqrt{\frac{4C_1 + C_3[\hat{n}_1^2 + \hat{n}_2(\hat{n}_2 + 2)]}{4C_1 + 2C_3\hat{n}_1(\hat{n}_2 + 1)}}, \qquad (90)$$

where $\dot{f}_1^{(1,1)}(\hat{n}_1, \hat{n}_2)$ is the solution of equation (20) with $\alpha = 0$ and $\dot{f}_1^{(1,1)}(\hat{n}_1, \hat{n}_2) = \dot{g}_1^{(1,1)}(\hat{n}_1, \hat{n}_2)$. Similar to the definition of nonlinear coherent state [31,32], $w(\hat{n}_1, \hat{n}_2)\exp[-\mathrm{i}(\phi_1 - \phi_2)]$ (see equation (88)) may be naturally called as the nonlinear phase difference operator, which in fact plays the role of amplifying the phase difference. Thus, equation (88) shows that the phase properties of $\mathcal{H}$ can be described by the nonlinear phase difference operator, while, as we know, the phase properties of the angular momentum system may be described by the phase difference operator of the two-dimensional harmonic oscillator [30].

Introduce another pair of Hermitian phase operators

$$\begin{aligned} \cos\Phi &= \tfrac{1}{2}(\mathcal{E}_- + \mathcal{E}_+), \\ \sin\Phi &= \tfrac{1}{2\mathrm{i}}(\mathcal{E}_- - \mathcal{E}_+), \end{aligned} \qquad (91)$$

it is easy to get

$$\begin{aligned} [\mathcal{J}_3,\ \cos\Phi] &= -\mathrm{i}\sin\Phi, \\ [\mathcal{J}_3,\ \sin\Phi] &= \mathrm{i}\cos\Phi, \end{aligned} \qquad (92)$$

which is similar to equation (85).

6. Conclusions

In this paper we have obtained the explicit expressions for two kinds of two-boson realizations of the Higgs algebra $\mathcal{H}$ by generalizing the well known Jordan–Schwinger realizations of SU(2) and SU(1,1). In each kind, the unitary realization, the (constrained) nonunitary realizations of the $(1, 1)$ case, and the properties of their respective acting spaces have been discussed in detail, together with the results of the $(2, 2)$ case. The other simple two-boson realizations for $k \neq l$, for example, $(k, l) = (1, 2)$, $(2, 1)$, etc., have also been obtained by solving equation (18) and (42), however, they are not given here because of their complex expressions. It is worth mentioning that for equation (16) in the first kind of realizations, its solution (17), which can be found its prototype for SU(2), is not unique, since, for example, it is determined up to any periodic function $\mathcal{T}(m)$ of an arbitrary but finite period m, namely, the constant α in equation (17) may be replaced by $\mathcal{T}(m)$, and for the $(1, 1)$ case the general solution of equation (16) should be $\hat{n}_1 + x(\hat{N})$, where $x(\hat{N})$ is an arbitrary function of $\hat{N}(= \hat{n}_1 + \hat{n}_2)$. Similar properties exist for equation (40) in the second kind of realizations. Furthermore, we have revealed the fact that the nonunitary realizations and the unitary ones may be related by the similarity transformations, which have been obtained by solving the corresponding unitarization equations satisfied by the nonunitary realizations. Finally, as applications, first we have found that the Kepler system in the two-dimensional curved space may be described by the dynamical group chain, $\mathcal{H} \supset SO(2)$, that is, there exists a simple relation between the Hamiltonian of this Kepler system and the Casimir operator of $\mathcal{H}$, and then obtained the energy levels by the eigenvalue of the Casimir invariant. Secondly, we have constructed the phase operators of the Higgs algebra in terms of the first kind of two-boson unitary realization, which hold the similar properties as the phase operators of the ordinary angular momentum systems. Due to the tight relations between boson operators and differential operators, for example, $a_i \leftrightarrow \partial_{x_i}$ ($i = 1, 2$) and $a_i^+ \leftrightarrow x_i$, the two-variable differential realizations of the Higgs algebra may be obtained directly from the above various two-boson realizations. The method adopted in this paper may be naturally generalized to the case of the multi-boson (or the deformed boson, the (deformed) fermion, etc.) and be used to treat the general PAMA given by equation (1).

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